Dynamic loadings on pedestrian bridges

Contents

Introduction

Pedestrian bridges are particularly susceptible to pedestrian-induced vibrations. The bridges tend to be slender and flexible structures with natural frequencies low enough to be excited by pedestrians, while people tend to be more aware of vibrations when they are on foot than they would be if they were in a motor vehicle.

The aim of this paper is briefly to cover the dynamic forces likely to be generated by walking, acceptable vibration levels in pedestrian bridges, what AS 5100.2 says about these, and what it does not say. The paper will then present a statistical approach to the dynamic loading from large numbers of pedestrians walking at random, and discuss non-random (synchronised) walking. It will also discuss the use of tuned mass dampers, before finishing with comments on an area where the author believes further research would be useful.

Dynamic loads imposed by a single person walking

In addition to the static force of his weight, a person imposes dynamic forces on the surface across which he is walking.

Vertical force

Associated with each foot-fall there is a complicated force-versus-time pattern, involving a heel strike, a roll forward, and an ankle-flick to prepare for the next step. Furthermore, since we are talking about walking rather than running, there is a time with both feet on the ground. The footfall frequency (*FF*) for "normal" walking can be anything in the range from 1.6 Hz to 2.4 Hz. Numerous researchers have investigated all this. Their results are usually presented as the Fourier components of the force-time pattern, with each component's amplitude expressed as a proportion of the pedestrian's weight (*W*).

Thus, typically, for a person walking in a natural and unencumbered way:

Component	First	Second	Third
Magnitude	$\pm 0.40 \times W$	$\pm 0.10 \times W$	$\pm 0.10 \times W$
Frequency	$1 \times FF$	$2 \times FF$	$3 \times FF$

It is quite common to ignore all except the primary component.

Lateral force

Except in the case of a tightrope walker, a pedestrian's centre of gravity moves slightly from side to side as he walks. Such a gait requires (through Newton's second law) an alternating lateral force, which (through Newton's third law) results in an opposite force being applied to the walking surface. Again, this force varies in a non-sinusoidal way, and is best expressed in terms of its first few significant Fourier components. Typically these might be:

Component	First	Second	Third
Magnitude	$\pm 0.05 \times W$	$\pm 0.04 \times W$	$\pm 0.01 \times W$
Frequency*	$0.5 \times FF$	$1.5 \times FF$	$2.5 \times FF$

Again, it is quite common to ignore all except the first component, but note that the second component can be nearly as large.

* The even-numbered harmonics do not appear here because nearly all research shows them to have much smaller amplitudes than the odd-numbered harmonics[1].

Longitudinal force

There are also varying longitudinal forces associated with walking, and these can be quite large. However with most structural systems forces in the longitudinal direction are stiffly resisted, and their dynamic contribution can be ignored.

Effect of crowding

These forces assume that our pedestrian is moving freely. If he is in a crowd he will be hemmed in to some extent. This will slow him down, resulting in a reduction in the dynamic forces his walking imposes on the bridge.

There are varying opinions on what level of crowding results in what level of force reduction. Some references[2,3] state that at pedestrian densities over 1.0 persons/m^2 freedom of movement is "greatly inhibited". At the other end of the spectrum, some others[4] suggest that the transition from uninhibited to inhibited might not occur until 1.8 or even 2.0 persons/m^2.

Acceptable vibration levels

The level of sustained harmonic vibration that a person finds "noticeable", "unpleasant" or "alarming" when traversing a pedestrian bridge varies between individuals. Unsurprisingly then, recommendations that appear in the literature on the subject embrace a considerable range of values. Similarly, different countries' design codes offer varying suggestions. What is generally agreed, however, is that over the sorts of vibration frequencies likely to be troublesome in pedestrian bridges, the lower the vibration frequency the larger the dynamic deflection people will tolerate. It is also agreed that different limits apply to vertical and lateral motions, with people being less tolerant of lateral motion.

When reading any rules and suggestions you need to note whether the limits are expressed as deflections or velocities or accelerations. For deflection-based limits you also need to know whether the limits are expressed as mathematical amplitudes or as peak-to-peak excursions.

Approach taken by AS 5100.2–2004

The Australian code takes a fairly minimalist approach to the dynamics of pedestrian bridges under pedestrian loadings. Its four short paragraphs on the subject can be boiled down to the following.

- You only need to investigate the problem if the bridge has a vertical vibration mode whose frequency lies between 1.5 Hz and 3.5 Hz, or a horizontal vibration mode whose frequency is less than 1.5 Hz.
- It states that you need to consider the dynamic forces from one solitary pedestrian.
- Vertical dynamic deflections should not exceed the value extracted from a chart the code provides. Typical values from this chart are ±3 mm at 1.5 Hz and ±1.4 mm at 3 Hz. (Note that, until corrected in 2010, this chart had a factor-of-ten error in the labelling of its vertical axis[5].)
- For this exercise, the "design pedestrian" weighs 700 N and walks at between 1.75 and 2.5 footfalls per second.
- There is a footnoted warning about the possibility of crowd loading, and the need for specialist assistance if this applies.
- The Commentary includes a warning about "synchronous lateral excitation", and a reference that can be used for assistance on this.

Matters that the code does not cover include the following.

- The vibration limits that should be applied to horizontal vibrations. AS 5100 is not alone in its shyness here. Among the few design codes giving recommendations we have:
  - Eurocode 1, which limits the horizontal vibrational acceleration (in m/s^2) to the lesser of $0.14\sqrt{(f_h)}$ and 0.15 (with f_h in Hz);
  - Eurocode 5, which limits it to 0.20 m/s^2.
- The determination of the likely dynamic forces that will result from multiple pedestrians, a subject to which we will now turn our attention.

Dynamic loads imposed by multiple independent pedestrians

For bridges which will only ever carry a small number of pedestrians simultaneously, it seems reasonable, and not unduly conservative, simply to assume that all of them are walking with their footfalls at the same frequency and phase. In military circles, this is called *marching*. Then the dynamic force from N pedestrians is simply N times the dynamic force from a single pedestrian.

Things become more interesting once we allow for a larger number of simultaneous pedestrians. To calculate the dynamic forces as if all the people were marching results in a gross over-estimate of the dynamic force. The people are walking <u>independently</u>. The phases of their footfalls are random, and the frequencies of their footfalls are also essentially random. This means that at any point in time there will be a group of people who are applying a maximum dynamic force to the bridge deck, and this group will be largely "countered" by a different group of people who are applying a minimum dynamic force.

"Largely" countered perhaps, but not exactly countered. The two groups of people will usually contain different numbers of people. How different? This problem is not well covered in the literature, but the book by Bachmann et.al.[6] states that: "If a Poisson distribution of arrivals is assumed, a magnification factor of *m* can be derived equivalent to the square root of the number of people on the bridge at any one time. This factor *m* is then applied to the response caused by a single pedestrian."

Confidence in this result can be drawn from Physics, in the area of Statistical Mechanics. It can easily be shown[7] that for a "one-dimensional random walk" of N steps the "root mean square" departure will be $\sqrt{N}$ steps (unintended wordplay in "random walk" noted and appreciated).

Thus the dynamic loading from N people walking independently is, on average, equivalent to that from $\sqrt{N}$ people marching.

But structures are not usually designed to resist the forces that apply "on average". They should be designed to resist the forces whose probability of occurrence is considerably lower than 0.5. A rational decision on the magnitude of the design force requires some knowledge of the statistical distribution of the net force.

When I was first wrestling with this problem (in 2003) my colleagues and I searched for material on this distribution. We could not find anything, not even an acknowledgement that the problem existed. So we had to roll up our sleeves and do our own theoretical work to derive the distribution. This involved some "heroic" assumptions, among which were:

- the pedestrians are all walking with the same footfall frequency;
- the dynamic force-vs-time curve for each pedestrian takes the form of a square-wave.

An outline of this derivation is presented in an Appendix below.

Subsequently (in 2007) Brand, Sanjayan and Sudbury[4] (BS&S) published the results of their investigations into this area. Their work used a completely different set of assumptions in its derivation. Yet, as shown below, there is an encouraging degree of agreement between the two sets of results: differences of less than 10% in the range of values most likely to be used in practice.

Proportion of time	*Equivalent no. of marching people*	
number will be exceeded	*My work*	*BS&S*
75 %	0.319 $\sqrt{N}$	0.536 $\sqrt{N}$
50 %	0.674 $\sqrt{N}$	0.833 $\sqrt{N}$
36.8 %		$\sqrt{N}$
31.7 %	$\sqrt{N}$	
25 %	1.150 $\sqrt{N}$	1.177 $\sqrt{N}$
10 %	**1.645 $\sqrt{N}$**	**1.517 $\sqrt{N}$**
5 %	1.960 $\sqrt{N}$	1.731 $\sqrt{N}$
2 %	2.326 $\sqrt{N}$	1.978 $\sqrt{N}$
1 %	2.576 $\sqrt{N}$	2.146 $\sqrt{N}$
0.5 %	2.807 $\sqrt{N}$	2.302 $\sqrt{N}$
0.1 %	3.271 $\sqrt{N}$	2.628 $\sqrt{N}$

Note that all four sources (Bachmann's, Statistical Mechanics, my work, and BS&S) have their results scaling according to $\sqrt{N}$.

Synchronous lateral excitation

Synchronous excitation (also known as *lock-in*) occurs when the pedestrians walking across a bridge begin to walk in time with the vibration of the bridge, and therefore with each other. Initially they do this subconsciously, because it makes the walking easier. As the vibration increases an increasing proportion of the pedestrians fall into step, leading to greater vibration amplitudes, leading to even greater pedestrian participation, leading to… . This is a classic positive feedback loop, which is eventually broken when a significant number of the synchronised pedestrians become alarmed by the resulting motion and stop walking. Synchronous excitation can occur for vertical motion as well as horizontal, but the vertical form is unlikely to occur for a well designed pedestrian bridge because the vibration limits usually imposed by codes of practice are smaller than the thresholds for the onset of synchronicity[2].

The phenomenon hit the headlines in 2000 with London's Millennium Bridge, and with Arup's subsequent theoretical and remedial work[8]. However it had been observed and reported previously, in 1993 by Fujino et.al.[9], and even earlier by some others.

With synchronous <u>lateral</u> excitation (SLE), there is also a second positive feedback effect at work. As individual pedestrians become increasingly aware of the lateral motion of the bridge deck, they widen their gait to increase their stability. This causes an increase in the lateral dynamic forces they individually apply to the bridge[10].

Various researchers have attempted to establish the vibration threshold for the onset of SLE. The loose consensus seems to be that SLE will begin when the amplitude of the lateral vibration amplitude is about ±1 to ±2 mm [ref 2], and that when the vibration amplitude reaches ±5 mm about 40% of the pedestrians will be locked in[2,8]. If a bridge has a natural frequency in the range 0.5 to 1.5 Hz, the best way to avoid SLE is to ensure that the bridge has sufficient damping to keep the vibration under this threshold when the crowd loading is random.

As stated above, if SLE does take hold the vibration will not increase without limit, because some people will stop walking. Leaving aside the fact that this probably constitutes a serviceability failure, how large a vibration will result? And how long will it take for this peak vibration to develop? Nakamura[11] has developed an empirical model to answer these questions, but his final result has no explicit analytical solution and requires the use of advanced numerical methods before it can be applied. To get around this, Niall[12] made a small change to Nakamura's model formulation, producing an explicit equation for the way the vibration amplitude develops.

Use of tuned mass dampers

One way in which additional damping can be incorporated into a structure is through the use of tuned mass dampers (TMDs). An excellent definition of these is given in Bachmann[6], and is paraphrased below.

> A tuned mass damper is a vibratory subsystem attached to a larger primary system. It consists in general of a mass, a spring and a damper. Accurate tuning of the frequency of the TMD results in induced inertia forces which counteract the external forces applied to the primary system, and less work is done on this system. Hence the normal practical function of the TMD is to reduce the resonant oscillations of the primary system. However, to achieve this, large oscillations must be accepted by the TMD.

Matters to be considered if designing TMDs include the following.

- A TMD is only effective over a narrow frequency band, to which it must be tuned. While a natural frequency that has been predicted by mathematical modelling will probably be adequate for preliminary sizing of the TMD, the final tuning must be done to the measured natural frequency of the as-constructed structure.
- The design should allow for re-tuning later in the life of the structure, in case of future changes in the structure's characteristics or usage.
- TMDs will not work satisfactorily for structures with several closely spaced natural frequencies which all tend to be excited by the dynamic action in question.
- A TMD should ideally be located where the vibration of the primary structure is greatest.
- The attachment of the TMD to the primary structure must be able to transmit the TMD's dynamic forces.
- The bridge designer must consider fatigue behaviour within the TMD, and within the primary structure in the vicinity of the TMD.
- The overall design should prevent the mass from falling if the spring fails. (In this regard, compression springs might be more suitable than tension springs.)
- Maintenance access to the TMD should be provided.
- The TMD must be surrounded by sufficient clearance to accommodate its own oscillations.

- TMDs are not cheap. If you can avoid them by designing a stiffer structure, do so. If you are not sure whether they will be required, and if you can accept a possible initial period of excessive vibrations, design for the inclusion of TMDs (supports, access, clearances, etc), but do not actually include them until the need for them is confirmed in operation.

Effect of large build-up time

The problem

[This section concerns an area that the author believes needs some original research. He is not aware of any work that has been done on it, and what follows is nothing more than a collection of his musings over several years.]

A bridge has large inertia, and the rate of energy input from our $X\sqrt{N}$ people "marching" across it is small. The peak predicted vibration develops gradually, over a period referred to as the *build-up time*. If the people are genuinely marching, then the peak will eventually be reached. But (unless we have some synchronicity) they are not marching: they are walking independently. The net number of coincident walkers (N_{eff}) will be changing continuously because, contrary to what was assumed above (in the derivation of the $X\sqrt{N}$ result), the footfall frequencies of the pedestrians are not all the same. The higher the value of N_{eff} at any point in time, the lower the likelihood that it will be sustained for long enough to allow the bridge's vibration to reach its theoretical peak value.

What form might research into this take, and in what form might the results best be presented to design engineers?

A precise definition of build-up time

When a harmonic loading is applied to a lightly damped structure that is initially at rest, the amplitude of the structure's resulting vibration increases at an ever-decreasing rate, asymptotically approaching the "steady-state" harmonic vibration. Thus, theoretically, the build-up time is infinite. This is of no use to us. To be usable, our definition of build-up time must be finite, unambiguous, and widely accepted.

It seems to the author that a suitable candidate for this definition would be the time taken for the structure's vibration amplitude to build up from zero to 90% of its final steady-state value[13].

Statistical distribution of footfall frequencies

The footfall frequencies of any group of arbitrarily selected people will have some sort of statistical distribution. This distribution should be relatively easy to establish by simple observation. One such group's distribution should be pretty much the same as any other group's, but consideration might need to be given to the possibility of "specialised" groups of people all using a bridge at the one time. A crowd of young people leaving a pop concert, for example.

For our present purposes it is the spread of the footfall frequency distribution that is important, rather than the actual values of the frequencies. Bearing this in mind, hopefully our intrepid researcher will be able to come up with a single distribution that is conservative but yet is still representative.

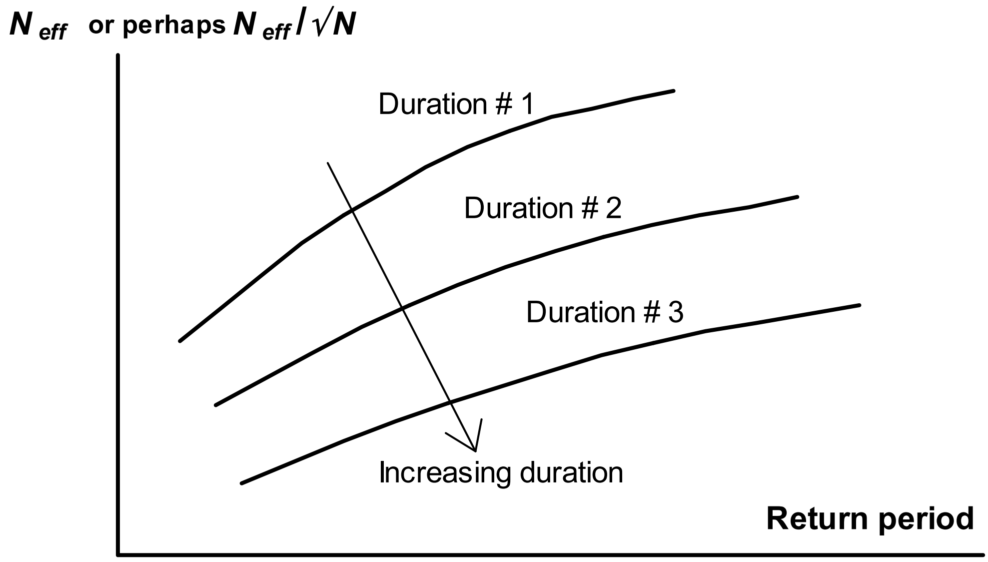
It seems intuitively obvious that with a narrow distribution N_{eff} will change only slowly, whereas with a broad distribution N_{eff} will change more rapidly. If so, a conservative distribution will be one that errs on the narrow side.

Statistical distribution of the durations of N_{eff}

For a given size of crowd (N), we are looking for the likelihood that the net number of coincident walkers (N_{eff}) will remain continuously at or above a given value for a given period of time. This problem can be investigated in one of three ways.

- By physical experimentation. Get a large platform supported on load cells, a large number of people, and an even larger number of pairs of shoes. Set the people walking endlessly, recording the resulting load-vs-time data continuously. An expensive approach, but it does at least have the advantage that it obviates the need for a separate experiment to establish the statistical distribution of footfall frequencies.
- Analytically. It might be possible to derive a formula for the result we want. This is the ideal approach if it can be done, but it is certainly beyond the mathematical abilities of this author.
- By Monte Carlo simulation. Inelegant, and a bit defeatist perhaps, but in the author's opinion it is the approach most likely to be adopted.

Whatever approach is used, the author believes that the best way to present the results would be as a set of charts, with the charts being in the form exemplified below.



Points to bear in mind when considering this chart:

- The likelihoods are expressed as return periods rather than straight probabilities. (These return periods might best be measured in hours, or even in minutes, rather than in years as is done in hydrology.)
- The return periods give a measure of the occurrence probabilities for that portion of the bridge's life when it is carrying N pedestrians. They do not apply to total lapsed time.
- At this stage we have to assume that (within reason) a different chart will be required for each different crowd size, ie for different N values.
- The investigations discussed earlier in this document have results that scale according to $\sqrt{N}$, suggesting the possibility that these new results might also scale that way. If so, then all crowd sizes will be able to be covered by the one, appropriately modified, chart. Hence the alternative label on the ordinal axis above.

- It is possible that separate charts might be required for vertical and horizontal forces. The author has not yet had time to think this through.

Bringing it all together

Once we have the research results outlined above, the design procedure for determining the dynamic load likely to be imposed by a large number of non-synchronised pedestrians becomes quite straight forward.

- Calculate the build-up time for the bridge (using the agreed standard definition).
- Decide on the appropriate return period for any resulting serviceability failure. (Standards bodies might need to provide some sort of guidance here.)
- Estimate the maximum likely number of freely moving pedestrians (N) that can fit on the bridge.
- Use a chart of the form shown above to arrive at the value of N_{eff} for the return period and for a duration related to the bridge's build-up time.
- The required dynamic load is now fully known.

Appendix A — Derivation of Random Walk formulae

CALCULATION SHEET

Project title		Project number	
Designed by	Date 23 May 2015	Document number	
Checked by	Date	Sheet 1	of 4

ONE-DIMENSIONAL RANDOM WALK

We wish to investigate a one-dimensional random walk. Our subject takes a walk of N steps. Before he takes each step he tosses a coin — on "heads" he steps eastward, on "tails" he steps westward. We wish to know how far he is likely to end up from his starting point. Note this "how far": we are indifferent to whether he ends up to the east or to the west, we just want to know the distance he ends up from his starting point.

Having said that, however, it is useful to begin by studying the case where we do draw a distinction between east and west.

LOCATION OF END POINT

If we call a step eastward a "pass" and a step westward a "fail" our situation is now a textbook case for the Binomial Distribution. Further, if we assume N to be large, we can use the Normal Distribution to ~~[illegible]~~ approximate the Binomial Distribution.

Let x be the walker's final position, measured as the net number of steps east of the starting point. Using the Normal approximation it is easy to show that the probability density function for x is

$$f(x) = \frac{1}{\sqrt{2\pi N}} e^{-x^2/(2N)}$$

The mean value for x is $\mu = \int_{-\infty}^{\infty} x . f(x)\,dx \Big/ \int_{-\infty}^{\infty} f(x)\,dx = 0$

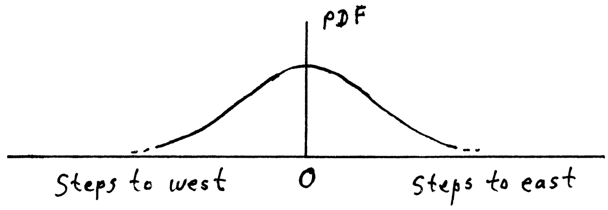
CALCULATION SHEET

Project title ______ Project number ______

Designed by ______ Date 23/5/15 Document number ______

Checked by ______ Date ______ Sheet 2 of 4

and the variance of x is $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \Big/ \int_{-\infty}^{\infty} f(x)\,dx = N$

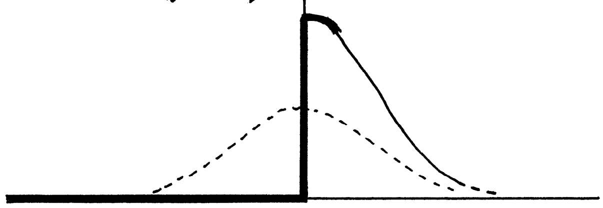


DISTANCE FROM START

How does this change when we consider instead the DISTANCE from the start point? In other words, we now want to investigate the distribution of $|x|$ rather than x.

Since $f(x)$ is symmetrical about the origin we can state that the distribution function for $|x|$ is

$$g(x) = \begin{cases} 0 & \text{for } x < 0 \\ 2f(x) & \text{for } x \geq 0 \end{cases}$$



The mean for $|x|$ is $\mu = \int_{-\infty}^{\infty} x \, . \, g(x) \, . \, dx \Big/ \int_{-\infty}^{\infty} g(x)\,dx$

$$= \int_{0}^{\infty} 2x \, . \, f(x)\,dx \Big/ 1 = \sqrt{\frac{2N}{\pi}}$$

CALCULATION SHEET

Project title ______ Project number ______

Designed by ______ Date 23/5/15 Document number ______

Checked by ______ Date ______ Sheet 3 of 4

and the variance is $$\sigma^2 = \int_0^{\infty} 2\left(x - \sqrt{\frac{2N}{\pi}}\right)^2 f(x)\,dx \Big/ 1 = \frac{\pi - 2}{\pi} N$$

CONFIDENCE LEVELS

The distribution $g(x)$ is very highly skewed, so the "usual" way of establishing confidence levels will not work, even approximately. We need to derive our own from first principles.

The probability of x being greater than X is given by

$$Prob(x > X) = \int_X^{\infty} g(x).dx = \int_X^{\infty} 2f(x)\,dx = 2 - 2\underbrace{\int_{-\infty}^{X} f(x)\,dx}_{*}$$

$$* \int_{-\infty}^{X} f(x)\,dx = \frac{1}{\sqrt{2\pi N}} \int_{-\infty}^{X} e^{-x^2/(2N)}\,dx$$

Substitute $z = x/\sqrt{N}$

Integral becomes $$\frac{1}{\sqrt{2\pi N}} \int_{-\infty}^{X/\sqrt{N}} e^{-z^2/2} . \sqrt{N} . dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X/\sqrt{N}} e^{-z^2/2}\,dz$$

So $$Prob(x > X) = 2 - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{X/\sqrt{N}} e^{-z^2/2}\,dz$$

~~Substitute~~ Put $u\sqrt{N}$ for X.

$$Prob(x > X) \equiv Prob(x > u\sqrt{N}) = 2 - \sqrt{\frac{2}{\pi}} \int_{-\infty}^{u} e^{-z^2/2}\,dz$$

CALCULATION SHEET

Project title ______ Project number ______

Designed by ______ Date 24/5/2015 Document number ______

Checked by ______ Date ______ Sheet 4 of 4

Use the "computer algebra" program DERIVE to evaluate this integral and establish a table of confidence levels. Results are:

Distance from start point to end point (measured in steps)	Probability of this distance being exceeded
0	1·0
$0{\cdot}01257\sqrt{N}$	0·99
$0{\cdot}06283\sqrt{N}$	0·95
$0{\cdot}12574\sqrt{N}$	0·90
$0{\cdot}31873\sqrt{N}$	0·75
$0{\cdot}67451\sqrt{N}$	0·50
$\sqrt{\frac{2}{\pi}}\cdot\sqrt{N} = 0{\cdot}79788\sqrt{N}$	0·42494
$1\sqrt{N}$	0·31731
1·15036	0·25
1·64486	0·10
1·95997	0·05
2·32635	0·02
2·57583	0·01
2·80703	0·005
3·29053	0·001

A Monte-Carlo style simulation of a one-dimensional random walk was developed in Excel. Its results were fully consistent with the above formulae and values.

Document revision history

Revision	*Comments*	*When released*	*By whom released*
0	Originally written by Robert Niall as a talk, which he delivered to the Victorian branch of the Concrete Institute of Australia on 15th February 2011.	15-Feb-2011	R.M.Niall
1	Various minor tweakings.	16-Nov-2012	R.M.Niall
2	Added derivations of Random Walk formulae.	24-May-2015	R.M.Niall

Notes & references

1 One can easily show, without resort to detailed mathematics, that if a finite signal trace (1) is anti-symmetric about its midpoint, and (2) consists of two halves each of which is symmetric about its own midpoint, then the Fourier decomposition of the trace will comprise only odd-harmonic sine curves.

2 International Federation for Structural Concrete (fib). 2005. *Guidelines for the design of footbridges*. Bulletin no. 32.

3 Bachmann, H. and W.J. Ammann. 1987. *Vibrations in structures induced by man and machines*. Structural Engineering Document 3e, International Association for Bridge and Structural Engineering (IABSE).

4 Brand, M., J.G. Sanjayan, and A. Sudbury. 2007. *Dynamic response of pedestrian bridges for random crowd-loading*. Australian Journal of Civil Engineering, Vol 3, No 1. Institution of Engineers, Australia.

5 The chart was corrected in Amendment 1, which was released in April 2010.

6 Bachmann, H. and others. 1995. *Vibration problems in structures — Practical guidelines*. Published by Birkhäuser Verlag, Basel.

7 See, for example, the URL http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/RandomWalk.htm [still accessible in November 2012].

8 Dallard, P., A.J. Fitzpatrick, A. Flint, S. Le Bourva, A. Low, R.M. Ridsdill Smith, and M. Willford. 2001. *The London Millennium Footbridge*. The Structural Engineer, Vol 79, No 22.

9 Fujino, Y., B.M. Pacheco, S. Nakamura, and W. Warnitchai. 1993. *Synchronisation of human walking observed during lateral vibration of a congested pedestrian bridge*. Earthquake Engineering and Structural Dynamics, Vol 22, pp 741–758.

10 Some more recent researchers are suggesting that this mechanism, the widening of the pedestrian's footsteps, is the predominant one, and that excessive lateral vibrations currently attributed to SLE can be explained without invoking SLE . See, for example: MacDonald, J.H.G. 2009. *Lateral excitation of bridges by balancing pedestrians*. Proceedings of The Royal Society A, Vol 465, No 2104, pp 1055–1073. (First published online, 16 Dec 2008.)

11 Nakamura, S. 2004. *Model for lateral excitation of footbridges by synchronous walking*. Journal of Structural Engineering, Vol 130, No 1, pp 32–37. ASCE.

12 Niall, R.M.. 2005. *Discussion of "Model for lateral excitation of footbridges by synchronous walking"*. Journal of Structural Engineering, Vol 131, No 7, pp 1150–1152. ASCE.

13 For a lightly damped dynamic system comprising only a single degree of freedom and excited at its resonant frequency, the vibration's displacement amplitude develops according to

$$\Delta(t) \quad = \quad \left(1 - e^{-\xi\omega t}\right)\frac{F_0}{2\xi k}\sin(\omega t)$$

where ξ is the damping ratio, F_0 is the amplitude of the harmonic exciting force, k is the stiffness, m is the mass, and $\omega=\sqrt{(k/m)}$ is the resonant angular frequency. The (90%) build-up time for this is $2.303/(\xi\omega)$, or $4.606\,m/c$ where c is the viscous damping constant.